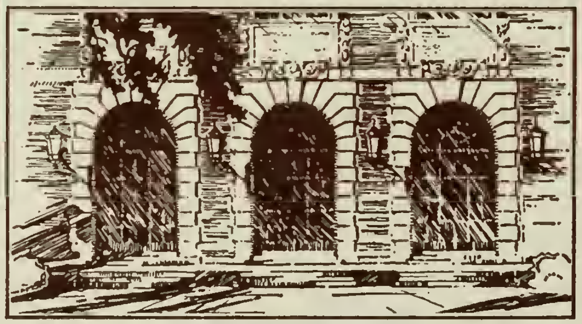

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STABILITY OF A SYSTEM OF EQUATIONS DESCRIBING THE FLOW BEHIND A SHOCK

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We shall investigate the stability of a solution of the following system of differential equations which arose from the investigation of shocked waves by Professor Taub:

$$\left.\begin{aligned}
&\frac{\partial \xi_i}{\partial T} = A_{ij}\left(\frac{\partial \xi_j}{\partial s} - C_j\right) \qquad i = 1,2\\
&\frac{\partial c}{\partial T} = \frac{\gamma - 1}{2}\, c\, \frac{\partial \xi_1}{\partial T}\\
&\frac{\partial m}{\partial T} = \frac{1}{m}\left(1 + \frac{\gamma - 1}{2} m^2\right)\frac{\partial \xi_1}{\partial T} - \frac{1}{c}\\
&\frac{\partial \lambda}{\partial T} = \cos\mu\, \frac{\partial \xi_2}{\partial s}\\
&\frac{\partial \alpha}{\partial T} = \frac{\sin\mu}{\lambda}\, \frac{\partial \xi_2}{\partial s}\quad .
\end{aligned}\right\} \tag{1}$$

Here and in the sequel we shall use the same notations as in a paper "Determination of Flows Behind Stationary and Pseudo-stationary Shocks" by Professor Taub.

To discuss the problem the most important coefficients in this system are A_{ij} $(i,j = 1,2)$ and they are given by (5.11):

$$(A_{ij}) = \frac{1}{\lambda(1 - m^2\cos^2\mu)}\begin{pmatrix} \sin\mu & m^2\cos\mu \\ \left(1 - \frac{1}{m^2}\right)\cos\mu & \sin\mu \end{pmatrix}. \tag{2}$$

The characteristic roots of this matrix are the solutions of

$$t^2 - \frac{2\sin\mu}{\lambda(1 - m^2\cos^2\mu)}\, t + \frac{1}{\lambda^2(1 - m^2\cos^2\mu)} = 0,$$

namely

$$\alpha_{\pm} = \frac{\sin\mu}{\lambda(1 - m^2\cos^2\mu)} \pm \frac{\cos\mu}{\lambda(1 - m^2\cos^2\mu)}\sqrt{m^2 - 1}\quad . \tag{3}$$

We may write our system of equations (1) in the following form:

$$\frac{\partial x_i}{\partial u} = a_{ij}\frac{\partial x_j}{\partial v} + b_i \qquad (i,j = 1,2,\ldots,6), \tag{4}$$

where a_{ij} and b_i are known functions of $x_1, \ldots, x_6$. Our case will be $x_1 = \xi_1$, $x_2 = \xi_2$, $x_3 = c$, $x_4 = m$, $x_5 = \lambda$, $x_6 = \alpha$, $u = T$, $v = s$ and

$$(a_{ij}) = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ \frac{\gamma-1}{2}c\ A_{11} & \frac{\gamma-1}{2}c\ A_{12} & 0 & 0 & 0 & 0 \\ -\frac{1}{m}(1+\frac{\gamma-1}{2}\ m^2)A_{11} & -\frac{1}{m}(1+\frac{\gamma-1}{2}\ m^2)A_{12} & 0 & 0 & 0 & 0 \\ 0 & \cos\mu & 0 & 0 & 0 & 0 \\ 0 & \frac{\sin\mu}{\lambda} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

$$(b_i) = \begin{bmatrix} -A_{1j}C_j \\ -A_{2j}C_j \\ -\frac{\gamma-1}{2}\ c\ A_{1j}C_j \\ \frac{1}{m}(1-\frac{\gamma-1}{2}\ m^2)\ A_{1j}C_j - \frac{1}{c} \\ 0 \\ 0 \end{bmatrix}.$$

The characteristics values of the matrix (a_{ij}) will be $\alpha_{\pm}$ and four 0's.

Suppose that there are variations δx_i of the solutions. Then these δx_i satisfy the equations:

$$\frac{\partial \delta x_i}{\partial u} = a_{ij}\frac{\partial \delta x_j}{\partial v} + \frac{\partial a_{ij}}{\partial x_k}\frac{\partial x_j}{\partial v}\delta x_k + \frac{\partial b_i}{\partial x_k}\delta x_k + \frac{\partial a_{ij}}{\partial x_k}\frac{\partial \delta x_j}{\partial v}\delta x_k \ . \tag{6}$$

We assume δx_i takes the form

$$\delta x_i = \delta x_i^o \, e^{\sqrt{-1}(\alpha u + \beta v)} \tag{7}$$

where α is real. Our system of equations (4) will be stable if the imaginary part of β is ≤ 0. We shall consider each x_i to be fixed, so that each coefficient a_{ij} (and b_i) is a constant, and α is sufficiently large, so that the magnitudes of $\frac{\partial a_{ij}}{\partial x_k} \frac{\partial x_j}{\partial v}$ and $\frac{\partial b_i}{\partial x_k}$ are negligible compared with α. Substituting (7) into (6), we get a system of linear equations between α and β with constant coefficients: namely

$$\alpha \delta x_i^o = \beta a_{ij} \delta x_j^o + \frac{\partial a_{ij}}{\partial x_k} \frac{\partial x_j}{\partial v} \delta x_k^o + \frac{\partial b_i}{\partial x_k} \delta x_k^o \; . \tag{8}$$

We have used an approximation that δx_k^o are so small that their square may be neglected. From our convention α is so large, and hence (8) may become

$$\alpha \delta x_i^o = \beta a_{ij} \delta x_j^o \; . \tag{8'}$$

Hence α/β must be one of characteristic values of the matrix (a_{ij}), i.e.

$$\beta = \frac{\alpha}{\alpha_{\pm}} \; . \tag{9}$$

In our case

$$\beta = \alpha\lambda(\sin\mu \mp \cos\mu \sqrt{m^2 - 1}) \; . \tag{9'}$$

Hence <u>the</u> <u>system</u> (1) <u>is</u> <u>stable</u> <u>if</u> <u>m $\geq$ 1</u>.
In case the flow is subsonic the investigation failed to produce any definite statement on the stability of equations.

In the following we shall discuss the similar problem for the difference equations derived from (1). Again we shall take our equations in the form (4).

Replace $\frac{\partial x_i}{\partial u}$ and $\frac{\partial x_i}{\partial v}$ by the difference quotients:

$$\frac{\partial x_i}{\partial u} = \frac{x_i(u + \Delta u, v) - x_i(u,v)}{\Delta u} \quad ,$$

$$\frac{\partial x_j}{\partial v} = \frac{x_j(u, v + \Delta v) - x_j(u,v)}{\Delta v} \quad ,$$

and use the notations such as

$$x_i^{n,\ell} = x_i(n\Delta u, \ell\Delta v).$$

Our basic difference equations will be

$$\frac{x_i^{n+1,\ell} - x_i^{n,\ell}}{\Delta u} = a_{ij} \; \frac{x_j^{n,\ell+1} - x_j^{n,\ell}}{\Delta v} + b_i . \tag{10}$$

We shall consider the variations δx_i of the form

$$x_i^{n,\ell} = \delta x_i^o s^n \xi^\ell, \qquad \text{where } s = e^{\sqrt{-1}\,\alpha\Delta u}, \quad \xi = e^{\sqrt{-1}\,\beta\Delta v}, \tag{11}$$

α is real and is to be assumed sufficiently large. The equations (10) will be stable if $|\xi| \leq 1$ for all (large) values of α. Substituting (11) into (10) we get

$$\delta x_i^{n+1,\ell} - \delta x_i^{n,\ell} = a_{ij}(\delta x_j^{n,\ell+1} - \delta x_j^{n,\ell}) \frac{\Delta u}{\Delta v} + \frac{\partial a_{ij}}{\partial x_k} \delta x_k^{n,\ell} (x_j^{n,\ell+1} - x_j^{n,\ell}) \frac{\Delta u}{\Delta v}$$

$$+ \frac{\partial a_{ij}}{\partial x_k} \delta x_k^{n,\ell} (\delta x_j^{n,\ell+1} - \delta x_j^{n,\ell}) \frac{\Delta u}{\Delta v} + \frac{\partial b_i}{\partial x_k} \; x_k^{n,\ell} \; \Delta u.$$

The principal term of the right hand side is the first one as $\Delta u, \Delta v \to 0$ keeping $\Delta u/\Delta v$ in a finite value. Hence again using (11)

$$\frac{\Delta v}{\Delta u}\frac{s-1}{\xi-1}\delta x_i^o = a_{ij}\delta x_j^o \;. \tag{12}$$

This relation may be obtained from (8') since (8') gives an approximation of variations when α is large. The relation (12) shows that $\frac{\Delta v}{\Delta u}\frac{s-1}{\xi-1}$ is a characteristic value of the matrix (a_{ij}):

$$\frac{\Delta v}{\Delta u}\;\frac{s-1}{\xi-1} = \alpha_{\pm}$$

The condition for the stability is $|\xi| \leqslant 1$ for all s with $|s| = 1$. Now $\xi = 1 + \frac{1}{\alpha_{\pm}}\frac{\Delta v}{\Delta u}(s-1)$, so that the above condition is satisfied if and only if $\alpha_{\pm} > 0$ and $\Delta v/\Delta u \leqslant \alpha_{\pm}$. Hence our difference equation is stable if $m \geqslant 1$ and

$$\frac{\Delta v}{\Delta u} \leqslant \frac{\sin\mu - \cos\mu\sqrt{m^2-1}}{\lambda(1 - m^2\cos^2\mu)} \tag{13}$$

The positiveness of the right hand side is guaranteed for a shock of non-zero strength, since in that case we have

$$1 - m^2\cos^2\mu = (\sin\mu + \cos\mu\sqrt{m^2-1})(\sin\mu - \cos\mu\sqrt{m^2-1}) > 0.$$

If the flow is subsonic: $m < 1$, then the system (10) of difference equations is unstable, i.e. the solution does not converge to a solution of our original equations (1) when $\Delta u, \Delta v \to 0$.

There are many ways to set up the difference equations corresponding differential equations. Integrating both sides of (4) with respect to u and applying the trapezoid rule to the right hand side, we get

$$x_i^{n+1} - x_i^n = \frac{\Delta u}{2}\left\{\left[a_{ij}\frac{\partial x_j}{\partial v}\right]^{n+1} + \left[a_{ij}\frac{\partial x_j}{\partial v}\right]^n\right\} + \frac{\Delta u}{2}(b_i^{n+1} - b_i^n), \tag{14}$$

where $x_i^n = x_i(n\Delta u, v)$, $b_i^n = b_i(x_1^n, \ldots, x_b^n)$ etc. Considering δx_i^n of the form:

$\delta x_i^n = \delta x_i^o s^n e^{v}$ where $s = e^{\sqrt{-1}\,\alpha u}$, we get a difference equation for δx_i such as

$$(s - 1)\,\delta x_i^o = \frac{\Delta u}{2}(X_{ij} + \beta Y_{ij})\delta x_j^o, \tag{15}$$

where
$$X_{ij} = \left[\frac{\partial b_i}{\partial x_j}\right]^{n+1} s + \left[\frac{\partial b_i}{\partial x_j}\right]^{n} + \left[\frac{\partial a_{ij}}{\partial x_j}\frac{\partial x_k}{\partial v}\right]^{n+1} s + \left[\frac{\partial a_{ik}}{\partial x_j}\frac{\partial x_k}{\partial v}\right]^{n},$$

$$Y_{ij} = \left[a_{ij}\right]^{n+1} s + \left[a_{ij}\right]^{n}.$$

In deriving (15) we assumed that δx_i^o are so small that their square may be neglected. The stability condition for the system (14) is that the real part of β is less than or equal to 0 whenever $|s| = 1$. In the actual form of (15) it seems difficult to give an explicit condition, so we assume here that Δu has been taken so small that we may replace (15) by

$$2(s - 1)\,\delta x_i^o \approx \beta \Delta u (s + 1) a_{ij} \delta x_j^o. \tag{15'}$$

If this approximation is permissible, then we can solve β as a function of s:

$$\beta = \frac{2}{\Delta u}\,\frac{1}{\alpha_{\pm}}\,\frac{s - 1}{s + 1},$$

where $\alpha_{\pm}$ are characteristic values of the matrix (a_{ij}). Since $|s| = 1$, $(s - 1)/(s + 1)$ is purely imaginary. In fact

$$\frac{s - 1}{s + 1} = \frac{i}{2}\,\frac{|1 - s|^2}{\tau(s)}.$$

Thus the real part of $\beta \leqslant 0$ for all $|s| = 1$, if and only if $\alpha_{\pm}$ are real, i.e. $m \geqslant 1$.

Summary

The system (1) of equations is stable if the flow is supersonic. I do not know whether stable or not in the other case. The similar stability conditions for the corresponding difference equations are examined for two kinds of systems. In both cases the systems are stable if the flow is supersonic, but unstable in case of subsonic flows.

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